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An Analytical Solution for Advection-Diffusion Equation with Advection–Dominated Term using Multiple-Scale Expansion Methods

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In this study, we solved the advection-diffusion sediment transport equation by taking into account the settling velocity and employing a multiple-scale expansion to approximate the dominant advection. We obtained an analytical solution in terms of the Green's function. The model results showed that, during one tidal cycle, TSS was concentrated at the surface and eventually transported to the sea. The pattern of TSS distribution was consistent with both constant and linear time-dependent river discharge. The simulation revealed that the concentration of TSS offshore was higher than near the estuaries, in agreement with our observations.

Keywords: *advection, Green's function, multiple-scale expansion, suspended sediment transport*

1. INTRODUCTION

The advection-diffusion equation constitutes a pivotal concept within applied mathematics, wherein it amalgamates two pivotal phenomena intrinsic to fluid dynamics and diffusion processes, specifically advection (convection) and diffusion (dispersion). The existence of this equation offers a mathematical underpinning for comprehending and scrutinizing diverse phenomena across various scientific domains, for instance, its utilization in modeling the transport of contaminants or chemical species (Dejak et al., 1987). Furthermore, it plays a pivotal role in the design of chemical processes, including chemical reactors and separation procedures (Wu et al., 2001), including oilspill (Eke et al., 2021). The numerical solution of this equation presents a formidable challenge, primarily due to its status as an open equation necessitating the specification of an advection field that satisfies the Navier-Stokes equations. Moreover, this equation is characterized by coefficients that are contingent upon independent variables, distinct from boundary conditions .

Finding an analytical solution to the advection-diffusion equation is a complex problem. This complexity arises from the combination of two different processes within the equation and the interactions between them. Nonlinearity, variable-dependent coefficients, diverse boundary conditions, and complex geometric shapes all contribute to the challenge of seeking analytical solutions. Analytical solutions for the advection-diffusion equation can be found only in a few exceptional cases. Perhaps the best-known analytical solution is the Gaussian solution (Carslaw & Jaeger, 1960).

The mouth of the estuary, with sediment dominated by mud, has an important cross-shore chenier dynamics effect (Colosimo et al., 2020, Tas et al., 2020), and sediment transport is dominated by the advection processes generated by the flow from the river. This condition is almost found in the tropical estuary, where the stratification is generally weak, so vertical mixing generates vertical transport (Mei et al., 1998 , Lissa and Stacey 2011, Mubarak et al., 2016). We show the role of the river's advection process and sediment supply in suspended sediment dynamics by constructing an analytical model and comparing it with measurement data. The model successfully explains the extent to which advection and river discharge affect the dispersion of suspended sediment in the estuary.

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2. METHOD

2.1 The Multiple Expansion Methods

We consider a two-dimensional advection-diffusion equation with the settling velocity taken into account as follows,

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+w_s\frac{\partial C}{\partial z}=D_x\frac{\partial^2 C}{\partial x^2} \tag{1}$$

where C is the concentration, w_s is the settling velocity, and D_x is the horizontal dispersion coefficient. We encountered a phenomenon where the advection effect is dominant, but the dispersion term cannot be simply ignored. Both effects are crucial, and they must be considered in the model. The multiple-scale expansion method may be used to handle this problem (Holmes 2021). Multi-scale analysis is a technique in perturbation theory that is applicable to systems with different timescale characteristics. For instance, consider an oscillator with a damping force where the damping effect is not noticeable on a short timescale, but the oscillatory motion ceases on a long timescale, indicating that the damping effect is apparent on a long timescale (Brakenhoff et al., 2020). This theory is based on the expansion of functions and coordinates in the small parameter ε. Due to the dispersion term being less dominant, we write Eq. (1) as,

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+w_s\frac{\partial C}{\partial z}=\varepsilon D_x\frac{\partial^2 C}{\partial x^2} \tag{2}$$

since there are fast-time and slow-time processes, we recognize these two time scales by introducing two time variables as fast and slow time respectively,

$$t_0=t \quad , \quad t_1=\varepsilon t. \tag{3}$$

the derivative operator becomes,

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}, \tag{4}$$

this yields,

$$\frac{\partial C}{\partial t_0}+\varepsilon\frac{\partial C}{\partial t_1}+u(x,t_0)\frac{\partial C}{\partial x}+\varepsilon u(x,t_1)\frac{\partial C}{\partial x}+w_s\frac{\partial C}{\partial z}=\varepsilon D_x\frac{\partial^2 C}{\partial x^2} \tag{5}$$

thus, we expand the solution as,

$$C(x,z,t)=C^{(0)}(x,z,t_0,t_1;\varepsilon)+\varepsilon C^{(1)}(x,z,t_0,t_1;\varepsilon)+\mathcal{O}(\varepsilon^2) \tag{6}$$

substituting Eq. (6) into Eq. (2), we get the equation in each order as follows,

$$\begin{aligned}&\frac{\partial C^{(0)}}{\partial t_0}+\varepsilon\frac{\partial C^{(1)}}{\partial t_0}+\varepsilon^2\frac{\partial C^{(2)}}{\partial t_0}+\ldots+\varepsilon\frac{\partial C^{(0)}}{\partial t_1}+\varepsilon^2\frac{\partial C^{(1)}}{\partial t_1}+\ldots+u(x,t_0)\frac{\partial C^{(0)}}{\partial x}\\&+\varepsilon u(x,t_0)\frac{\partial C^{(1)}}{\partial x}++\varepsilon^2 u(x,t_0)\frac{\partial C^{(2)}}{\partial x}+\ldots+\varepsilon u(x,t_1)\frac{\partial C^{(0)}}{\partial x}+\varepsilon^2 u(x,t_1)\frac{\partial C^{(1)}}{\partial x}\\&+\varepsilon^2 u(x,t_1)\frac{\partial C^{(1)}}{\partial x}+\ldots+w_s\frac{\partial C^{(0)}}{\partial z}+\varepsilon w_s\frac{\partial C^{(1)}}{\partial z}+\ldots=\varepsilon D_x\frac{\partial^2 C^{(0)}}{\partial x^2}+\varepsilon^2 D_x\frac{\partial^2 C^{(1)}}{\partial x^2}+\ldots\end{aligned} \tag{7}$$

collecting coefficients of equal powers of ε gives,

zero order,

$$\frac{\partial C^{(0)}}{\partial t_0}+u(x,t_0)\frac{\partial C^{(0)}}{\partial x}+w_s\frac{\partial C^{(0)}}{\partial z}=0 \tag{8}$$

first order,

$$\frac{\partial C^{(1)}}{\partial t_0}+u(x,t_0)\frac{\partial C^{(1)}}{\partial x}+w_s\frac{\partial C^{(1)}}{\partial z}=D_x\frac{\partial^2 C^{(0)}}{\partial x^2}-\frac{\partial C^{(0)}}{\partial t_1}-u(x,t_1)\frac{\partial C^{(0)}}{\partial x} \tag{9}$$

second order,

$$\frac{\partial C^{(2)}}{\partial t_0}+u(x,t_0)\frac{\partial C^{(2)}}{\partial x}+w_s\frac{\partial C^{(2)}}{\partial z}=D_x\frac{\partial^2 C^{(1)}}{\partial x^2}-\frac{\partial C^{(1)}}{\partial t_1}-2u(x,t_1)\frac{\partial C^{(1)}}{\partial x} \tag{10}$$

by using tidal velocity, then Eq. (8) becomes,

$$\frac{\partial C^{(0)}}{\partial t_0}+U\left(x,t_0\right)\frac{\partial C^{(0)}}{\partial x}+w_s\frac{\partial C^{(0)}}{\partial z}=0 \tag{11}$$

because the variable coefficient depends on χ and t_0, we can use the separation variable as,

$$C^{(0)}\left(t_0,x,\zeta\right)=\phi^{(0)}\left(t_0,x\right)Z^{(0)}\left(\zeta\right) \tag{12}$$

these yields,

$$\frac{1}{\phi^{(0)}\left(t_0,x\right)}\left[\frac{\partial\phi^{(0)}\left(t_0,x\right)}{\partial t_0}+U\left(x,t_0\right)\frac{\partial\phi^{(0)}\left(t_0,x\right)}{\partial x}\right]+\frac{w_s}{Z^{(0)}\left(z\right)}\frac{dZ^{(0)}\left(z\right)}{dz}=0 \tag{13}$$

the solution of the second term is given by,

$$Z^{(0)}\left(z\right)=exp\left[\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(\varsigma\right)}d\varsigma\right] \tag{14}$$

where Λ is a constant. Thus, Eq. (13) can be written as,

$$\frac{\partial\phi^{(0)}\left(t_0,x\right)}{\partial t_0}+U\left(x,t_0\right)\frac{\partial\phi^{(0)}\left(t_0,x\right)}{\partial x}+\Lambda\phi^{(0)}\left(t_0,x\right)=0 \tag{15}$$

assuming the solution in terms of a traveling wave $f(x,t_0)=f(kx-\omega t_0)=f(\xi)$, then we arrive,

$$\left[U\left(\xi\right)-\omega\right]\frac{d\phi^{(0)}\left(\xi\right)}{d\xi}+\Lambda\phi^{(0)}\left(\xi\right)=0 \tag{16}$$

thus, the solution is given by,

$$\phi^{(0)}\left(\xi\right)=K exp\left[-\int_{0}^{\xi}\left(\frac{\Lambda}{U\left(\xi'\right)-\omega}\right)d\xi'\right] \tag{17}$$

where K is the integration constant. Thus, the complete solution of zero order as,

$$C^{(0)}\left(t_0,x,z\right)=A_0 exp\left[\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(\zeta'\right)}d\zeta'-\int_{0}^{\xi}\left(\frac{\Lambda}{U\left(\xi'\right)-\omega}\right)d\xi'\right] \tag{18}$$

further, the first order can be written as follows,

$$\frac{\partial C^{(1)}}{\partial t_0}+u(x,t_0)\frac{\partial C^{(1)}}{\partial x}+w_s\frac{\partial C^{(1)}}{\partial z}=f\left(t_0,x,z\right) \tag{19}$$

where,

$$f\left(t_0,x,z\right)=D_x\frac{\partial^2 C^{(0)}}{\partial x^2}-\frac{\partial C^{(0)}}{\partial t_1}-u(x,t_1)\frac{\partial C^{(0)}}{\partial x} \tag{20}$$

2.2 Green's Function Solution

The solution of Eq. (19) can be obtained through the Green's function as follows,

$$C^{(1)}\left(t_0,x,z\right)=\iiint G\left(t_0,x,z;t_0',x',z'\right)f\left(t_0',x',z'\right)dt_0'dx'dz' \tag{21}$$

where the Green's function satisfies,

$$\left[\frac{\partial}{\partial t_0}+u(x,t_0)\frac{\partial}{\partial x}+w_s\frac{\partial}{\partial z}\right]G\left(t_0,x,z,t_0',x',z'\right)=-4\pi\delta\left(t_0-t_0'\right)\delta\left(x-x'\right)\delta\left(z-z'\right) \tag{22}$$

we see that the homogeneous part of Eq. (18) is similar to Eq. (10), so the Green's function will be constructed from the solution of Eq. (10). We choose the Green's function as,

$$G\left(t_0,x,z,t_0',x',z'\right)=\frac{1}{2\sqrt{\pi}}exp\left[\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(z-z'\right)}dz'-\int_0^{t_0}\int_0^{\chi}\left(\frac{\Lambda}{U\left(x-x',t_0-t_0'\right)-\omega}\right)dx'dt_0'\right] \tag{23}$$

with the Green's function above, the solution Eq. (19) becomes an integral problem of Eq. (20). This yields,

$$C^{(1)}\left(t_0,x,z\right)=\frac{A_0}{2\sqrt{\pi}}\int_{-\zeta}^{0}exp\left(\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(z-z'\right)}d\zeta'+\int_{-\zeta'}^{0}\frac{\Lambda}{w_s\left(z''\right)}dz''\right)d\zeta'\times\left[\hat{G}_1\left(t_0,x\right)+\hat{G}_2\left(t_0,x\right)\right] \tag{24}$$

where G_1 and G_2 will be determined by a letter.

The solution with the Green's function is a problem of calculating the convolution integral. First, substituting Eq. (18) into Eq. (20) yields,

$$f\left(t_0,\chi,\zeta\right)=A_0 exp\left[\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(\zeta'\right)}d\zeta'\right]\left(D_x\frac{\partial^2\hat{F}(t_0,\chi)}{\partial\chi^2}-u(\chi,t_1)\frac{\partial\hat{F}(t_0,\chi)}{\partial\chi}\right) \tag{25}$$

where,

$$\hat{F}(t_0,\chi)=exp\left[-\int_0^{t_0}\int_0^{\chi}\left(\frac{\Lambda}{U\left(\chi,t_0\right)-\omega}\right)d\chi'dt_0'\right] \tag{26}$$

we note that a term containing the derivative t_1 will result in a second-order divisor in ε, which will have a small value, so that we ignore it. Substituting (25) and Eq. (23) into Eq. (21), we arrive,

$$C^{(1)}\left(t_0,\chi,\zeta\right)=\frac{A_0}{2\sqrt{\pi}}\int_{-\zeta}^{0}exp\left(\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(\zeta-\zeta'\right)}d\zeta'+\int_{-\zeta'}^{0}\frac{\Lambda}{w_s\left(\zeta''\right)}d\zeta''\right)d\zeta'\times\left[\hat{G}_1\left(t_0,\chi\right)+\hat{G}_2\left(t_0,\chi\right)\right] \tag{27}$$

where,

$$\hat{G}_1\left(t_0,x\right)=D_x\int_0^{t_0}\int_0^{x}\left\{exp\left[-\int_0^{t_0'}\int_0^{x'}\left(\frac{\Lambda}{U\left(x'-x'',t_0'-t_0''\right)-\omega}\right)dt_0''dx''\right]\frac{\partial^2\hat{F}(t_0',x')}{\partial x'^2}\right\}dt_0'dx' \tag{28}$$

$$\hat{G}_2\left(t_0,x\right)=-\int_0^{t_0}\int_0^{x}U\left(x',t_1\right)\left\{exp\left[-\int_0^{t_0'}\int_0^{x'}\left(\frac{\Lambda}{U\left(x'-x'',t_0-t_0'\right)-\omega}\right)dt_0''dx''\right]\frac{\partial\hat{F}(t_0',x')}{\partial x'}\right\}dt_0'dx' \quad (29)$$

first, we calculate the derivation of $\hat{F}(t_0',x')$ as follows,

$$\frac{\partial\hat{F}(t_0',x')}{\partial x'}=-e^{-\hat{\Lambda}(t_0',x')}\frac{\partial\hat{\Lambda}\left(t_0',x'\right)}{\partial x'}=-e^{-\hat{\Lambda}(t_0',x')}\left[\int_0^{t_0'}\frac{\Lambda}{U\left(x',t_0''\right)-\omega}dt_0''\right] \quad (30)$$

where,

$$\hat{\Lambda}\left(t_0',x'\right)=\int_0^{t_0'}\int_0^{x'}\left(\frac{\Lambda}{U\left(x'',t_0''\right)-\omega}\right)dx''dt_0'' \quad (31)$$

and we have used Leibniz's rule integration formula $\frac{d}{dx}\int_0^{g(x)}f(t)dt=f(g)g'(x)$. The second derivative,

$$\frac{\partial^2\hat{F}(t_0',x')}{\partial x'^2}=-e^{-\hat{\Lambda}(t_0',x')}\left[\left(\int_0^{t_0'}\frac{\Lambda}{U\left(x',t_0''\right)-\omega}dt_0''\right)^2-\int_0^{t_0'}\frac{\Lambda}{U\left(x',t_0''\right)-\omega}dt_0''\right] \quad (32)$$

substituting Eq.(32) and Eq.(30) into Eq.(28) and Eq.(29), respectively, we get the complete solution of Eq.(27) which is explicitly dependent on the form of *U(x,t)*. For instance *U(x,t)* is given by,

$$U\left(x',t_0''\right)=A_1cos\left(x',t_0''\right)-A_2\varepsilon cos2\left(x',t_0''\right)-A_3\varepsilon^2x'sin2\left(x',t_0''\right) \quad (33)$$

where,

$$\left(x',t_0''\right)=kx'-\omega t_0'',\qquad A_1=\frac{gA}{c},\qquad A_2\varepsilon=\frac{g^2A^2}{8c^3},\qquad A_3\varepsilon^2=\frac{3g^2A^2\omega}{4c^4} \quad (34)$$

we use Maclaurin's series ($\frac{1}{x}=\sum_{n=0}^{\infty}(1-x)^n$) and keep it up to second order. Then we get,

$$\begin{aligned}&\left[A_1cos\left(x',t_0''\right)-A_2\varepsilon cos2\left(x',t_0''\right)-A_3\varepsilon^2x'sin2\left(x',t_0''\right)-\omega\right]^{-1}\approx\\&3A_1cos\left(x',t_0''\right)+2\omega A_1cos\left(x',t_0''\right)+4\omega+A_1^2cos^2\left(x',t_0''\right)+3A_2\varepsilon cos2\left(x',t_0''\right)\\&+2\omega A_2\varepsilon cos2\left(x',t_0''\right)+2A_1A_2\varepsilon cos\left(x',t_0''\right)cos2\left(x',t_0''\right)\\&+A_2^2\varepsilon^2cos^2 2\left(x',t_0''\right)+2A_1A_3\varepsilon^2cos\left(x',t_0''\right)x'sin2\left(x',t_0''\right)+3A_3\varepsilon^2x'sin2\left(x',t_0''\right)\end{aligned} \quad (35)$$

thus, we obtained,

$$\frac{\partial\hat{F}(t_0',x')}{\partial x'}=-e^{-\hat{\Lambda}(t_0',x')}\bar{F}(t_0',x');\qquad\frac{\partial^2\hat{F}(t_0',x')}{\partial x'^2}=-e^{-\hat{\Lambda}(t_0',x')}\left[\bar{F}^2(t_0',x')-\partial_{x'}\bar{F}(t_0',x')\right] \quad (36)$$

where,

$$\begin{aligned}\bar{F}\left(t_0',x'\right)=&\left[\frac{3A_1}{\omega}sin(t_0',x')+2A_1sin(t_0',x')+4\omega t_0'+\frac{A_1^2}{2}\left(t_0'+\frac{1}{2\omega}sin2(t_0',x')\right)+\frac{3A_2\varepsilon}{\omega}sin2(t_0',x')\right.\\&+2A_2\varepsilon sin2(t_0',x')+\frac{2A_1A_2\varepsilon}{\omega}\left(sin(t_0',x')cos2(t_0',x')+\frac{4}{3}sin^3(t_0',x')\right)+A_2^2\varepsilon^2\left(\frac{t_0'}{2}+\frac{1}{8\omega}sin4(t_0',x')\right)\\&\left.-A_1A_2\varepsilon^2\left(\frac{1}{3\omega}cos3(t_0',x')+\frac{1}{\omega}cos(t_0',x')\right)-\frac{3A_3x'\varepsilon^2}{2\omega}cos2(t_0',x')\right]\end{aligned} \quad (37)$$

and,

$$\frac{\hat{\Lambda}\left(t_0',x'\right)}{\Lambda} = -\frac{3A_1}{k\omega}cos(t_0',x') - \frac{2A_1}{k}sin(t_0',x') - \frac{2A_1^2}{8k\omega}cos2(t_0',x') - \frac{3A_2\varepsilon}{2k\omega}cos2(t_0',x') - \frac{A_2\varepsilon}{k}cos2(t_0',x')$$
$$+\frac{A_1A_2\varepsilon}{k\omega}\left(-cos(t_0',x') - \frac{1}{3}cos3(t_0',x') + \frac{4}{9}cos^3(t_0',x')\right) - \frac{A_2^2\varepsilon^2}{32k\omega}cos4(t_0',x') + \frac{1}{2}\left(A_1^2 + A_2^2\varepsilon^2\right)t_0'x' \tag{38}$$
$$-A_1A_2\varepsilon^2\left(\frac{1}{9k\omega}sin3(t_0',x') + \frac{1}{k\omega}cos(t_0',x')\right) - \frac{3A_3x'\varepsilon^2}{4k\omega}cos2(t_0',x')$$

by using an approximation $e^x \sim 1 + x + \frac{1}{2!}x^2 + \cdots$, and solving the integration with respect to (χ', t_0'), in the Green's function Eq. (23), this leads,

$$exp\left[-\int_0^{t_0'}\int_0^{x'}\left(\frac{\Lambda}{U\left(x-x'',t_0-t_0'\right)-\omega}\right)dt_0''dx''\right] \approx \Lambda - \frac{\left(3A_1+2\omega A_1\right)\Lambda}{k\omega}cos\left(x-x',t_0-t_0'\right) + 2\omega\Lambda x't_0'$$
$$+\frac{A_1^2}{4}\Lambda\left((x'-t_0')^2 - \frac{1}{2k\omega}cos2\left(x-x',t_0-t_0'\right)\right) - \frac{\left(3A_2+2\omega A_2\right)\Lambda\varepsilon}{4k\omega}cos2\left(x-x',t_0-t_0'\right) + \ldots \tag{39}$$

further, substituting Eq.(39), Eq. (38), Eq.(37), and Eq. (36) into Eq. (28) and Eq.(29), yields,

$$\frac{\hat{G}_1\left(t_0,x\right)}{D_x} = \iint_A dx'dt_0' + \iint_B dx'dt_0' + \iint_C dx'dt_0' + \iint_D dx'dt_0' + \ldots$$
$$+\varepsilon\left[\iint_I dx'dt_0' + \iint_{II} dx'dt_0' + \iint_{III} dx'dt_0' + \iint_{IV} dx'dt_0' + \ldots\right] \tag{40}$$

where,

$$\iint_A dx'dt_0' = \frac{27A_1^4(1+2\omega)\Lambda^2}{k^2\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2\left(x',t_0'\right)cos\left(x',t_0'\right)dx'dt_0' \tag{41}$$

$$\iint_B dx'dt_0' = \frac{18A_1^4(1+2\omega)\Lambda^2}{k^2\omega^2}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^3\left(x',t_0'\right)dx'dt_0' \tag{42}$$

$$\iint_C dx'dt_0' = \frac{12A_1^4(3+2\omega)\Lambda^2}{k^2\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2\left(x',t_0'\right)cos2\left(x',t_0'\right)dx'dt_0' \tag{43}$$

$$\iint_D dx'dt_0' = \frac{9A_1^4(3+2\omega)\Lambda^2}{k^2\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2 2\left(x',t_0'\right)cos2\left(x',t_0'\right)dx'dt_0' \tag{44}$$

$$\iint_I dx'dt_0' = \frac{27A_1^2A_2(3+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2 2\left(x',t_0'\right)cos2\left(x',t_0'\right)dx'dt_0' \tag{45}$$

$$\iint_{II} dx'dt_0' = \frac{18A_1^2A_2(3+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^3 2\left(x',t_0'\right)cos2\left(x',t_0'\right)dx'dt_0' \tag{46}$$

$$\iint_{III} dx'dt_0' = \frac{12A_1^2A_2(1+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2 2\left(x',t_0'\right)\sin\left(x',t_0'\right)dx'dt_0' \tag{47}$$

$$\iint_{IV} dx'dt_0' = \frac{9A_1^2A_2(3+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2 2\left(x',t_0'\right)sin2\left(x',t_0'\right)dx'dt_0' \tag{48}$$

The other terms contain multiplication by the squared or higher, where this term has a divisor of c3, so this can be seen as a small perturbation, and it can be ignored. By using the same procedure, we arrive,

$$\hat{G}_2\left(t_0,x\right)=\iint_{A'} dx'dt_0' + \iint_{B'} dx'dt_0' + \iint_{C'} dx'dt_0' + ... + \varepsilon\left[\iint_{I'} dx'dt_0' + \iint_{II'} dx'dt_0' + \iint_{III'} dx'dt_0' + ...\right] \quad (49)$$

where,

$$\iint_{A'} dx'dt_0' = \frac{A_1^3(1+2\omega)\Lambda^2}{k\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin\left(x',t_0'\right)cos\left(x',t_0'\right)dx'dt_0' \quad (50)$$

$$\iint_{B'} dx'dt_0' = \frac{A_1^3(1+2\omega)\Lambda^2}{k\omega^3}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin\left(x',t_0'\right)cos^2\left(x',t_0'\right)dx'dt_0' \quad (51)$$

$$\iint_{C'} dx'dt_0' = \frac{6A_1^3(1+2\omega)\Lambda^2}{k\omega}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2\left(x',t_0'\right)cos\left(x',t_0'\right)dx'dt_0' \quad (52)$$

$$\iint_{I'} dx'dt_0' = \frac{6A_1^2A_2(3+2\omega)\Lambda^2}{k\omega}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin\left(x',t_0'\right)cos2\left(x',t_0'\right)dx'dt_0' \quad (53)$$

$$\iint_{II'} dx'dt_0' = -\frac{3A_1^2A_2(3+2\omega)\Lambda^2}{k\omega}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin\left(x',t_0'\right)cos\left(x',t_0'\right)cos2\left(x',t_0'\right)dx'dt_0' \quad (54)$$

$$\iint_{III'} dx'dt_0' = -\frac{6A_1^3A_2(3+2\omega)\Lambda^2}{k\omega}\int_0^x\int_0^{t_0} cos\left(x-x';t_0-t_0'\right)sin^2\left(x',t_0'\right)cos\left(x',t_0'\right)cos2\left(x',t_0'\right)dx'dt_0' \quad (55)$$

Finally, we use numerical integration to solve the convolution integral above. In this study, we limit it to first-order because second and higher-order disturbances will only give minor disturbances to our primary system.

3. RESULTS AND DISCUSSION

To demonstrate how the calculation we discussed earlier works, we use it to solve the issue of suspended sediment transport at the entrance of a river estuary. We take the example of an estuary located in the equatorial region, known as the Dumai estuary on the island of Sumatra, situated around the waters of the Singapore Strait. This estuary is characterized by the prevalence of fine and very fine sediments, primarily transported by tides (Alkhatib et al., 2007). In this area, freshwater mixes with saltwater, resulting in a slow river current. As a result, the suspended sediment from the upstream basin settles at the bottom, creating shallow bathymetry. Because this is not a journal about estuaries, we do not provide data related to processes that occur in estuaries; instead, we compare our calculation of previous results with total sediment transport (TSS) measurement data in the study area.

The advection processes in the estuary are governed by tidal currents. The relationship between tidal currents and sea level elevation for one-dimensional cases fulfills the following equation (Officer, 1978, Vongvisessomjai and Chatanantavet 2006),

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial \eta}{\partial x} \quad (56)$$

$$\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+\left(H+\eta\right)\frac{\partial u}{\partial x}=0 \quad (57)$$

where u is the zonal velocity, η is the tidal elevation, H is the water depth, and g is the gravitational acceleration. The equations are nonlinear, so only a few simple cases have analytical solutions. We use an approximation, i.e., expand the tidal current velocity (u) and the surface elevation (η) in $\varepsilon=H/\lambda$ with λ as the horizontal length scale. Because of $H<<\lambda$, the parameter $\varepsilon<<1$ is a small parameter. With this expansion, considering the Dirichlet boundary conditions, we get the elevation and tidal currents as follows,

$$\eta(x,t) = A\cos(kx-\omega t) - \frac{3gA^2\omega}{4c^3}x\sin 2(kx-\omega t) \tag{58}$$

$$u(x,t) = \frac{gA}{c}\cos(kx-\omega t) - \frac{g^2A^2}{8c^3}\cos 2(kx-\omega t) - \frac{3g^2A^2\omega}{4c^4}x\sin 2(kx-\omega t) \tag{59}$$

where $c=\sqrt{(gH)}$. Tidal data showed that the directional elevation occurred twice a day (semi-diurnal), with the highest and lowest tides in the Dumai estuary being 1.26 and 0.47 m, respectively. Also, the Formzhal number is about 0.239, indicating that the tidal type in Dumai estuary waters was a mixed directional type, tending to double a day (Amiruddin et al., 2011, Rifardi et al., 2020). The tidal current represented by Eq. (59) is depicted in Figure 1.

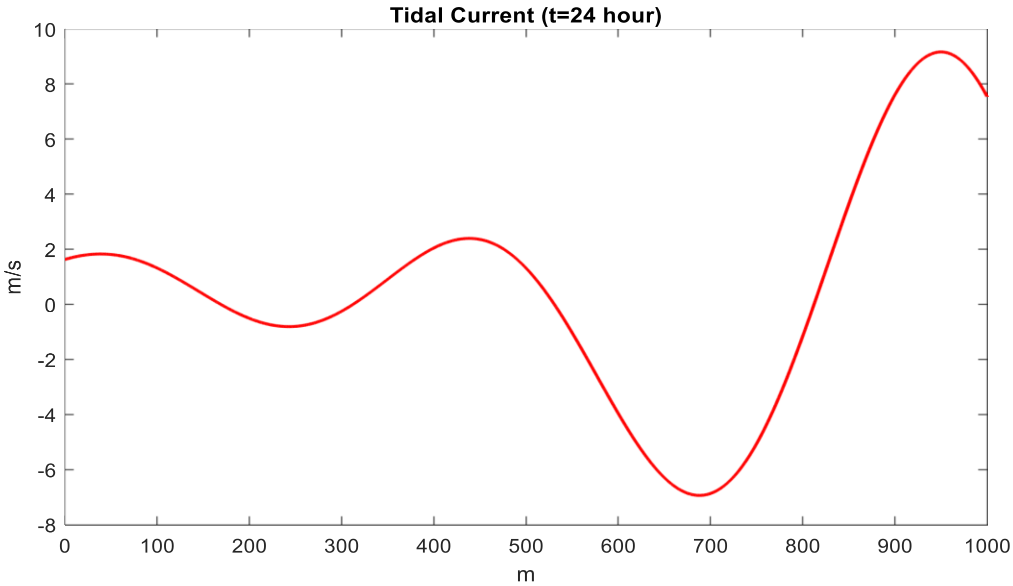

Figure 1. Tidal current for one tidal cycle period (0-1000 is mouth of estuary to offshore).

We simulate TSS dispersion based on Eq. (5), Eq. (18), and Eq. (27). The simulation was carried out for one tidal cycle for 24 hours, that is, there were two high tides and twice ebb. In this simulation, we performed a deposition velocity of 0.25 m/s and a horizontal dispersion coefficient of 0.005 m/s^2, TSS sources of 800 mgL^{-1} from the river spread to the surface of the open sea. At the first ebb, TSS with a concentration of about 60 mgL^{-1} spreads as far as 600 m with downward dispersion as far as 0.5 m from the surface. At the time, the first tide's highest concentration of 160 mgL^{-1} was 900 m away. At low tide, both concentrations spread more widely, with a concentration of 180 mgL^{-1}, and on the second tide, the concentration reaches 300 mg/L at a distance of about 1 km.

This study elucidates the behavior of TSS from rivers, illustrating that their dispersion and concentration significantly alter throughout the tidal cycle, with pronounced horizontal and vertical distribution patterns. These findings corroborate the earlier research conducted by Wang (2002), which was grounded in an idealized model. In the context of the Dumai estuary, particularly within the Mesjid estuary, a notable phenomenon of shallowing is observed, which is linked to the distribution of TSS from the river. This trend is anticipated to culminate in the formation of a sandbank in the forthcoming decades (Rifardi and Badrun, 2017). Furthermore, it is imperative to acknowledge the critical role of stratification in sediment transport dynamics within estuarine environments (Wang et al., 2015), a factor that has not been addressed in this paper and constitutes a notable limitation. Future research will encompass calculations pertaining to stratification to enhance the understanding of these processes.

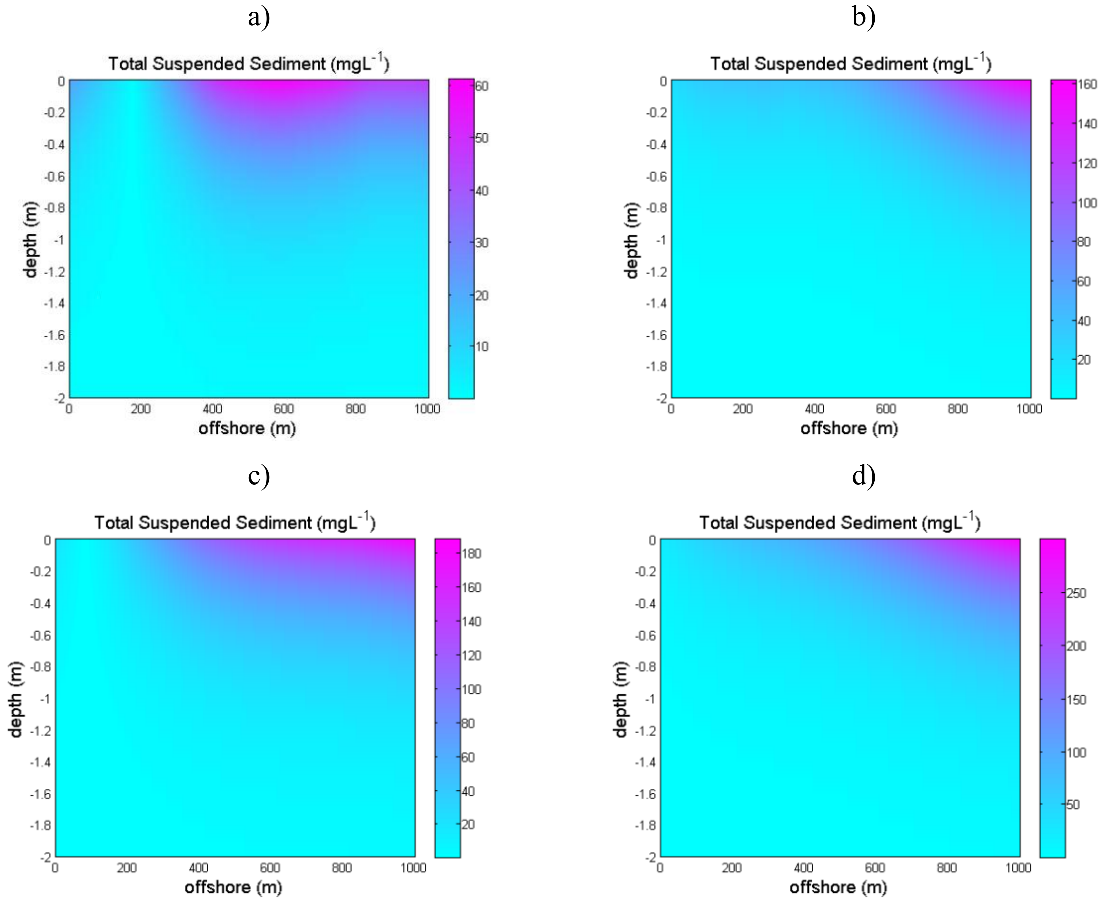

Figure 2. Suspended sediment transport into the mouth of the Dumai estuary over one tidal cycle period in the case of constant river discharge, a) first low tide, b) first high tide, c) second low tide, and d) second high tide.

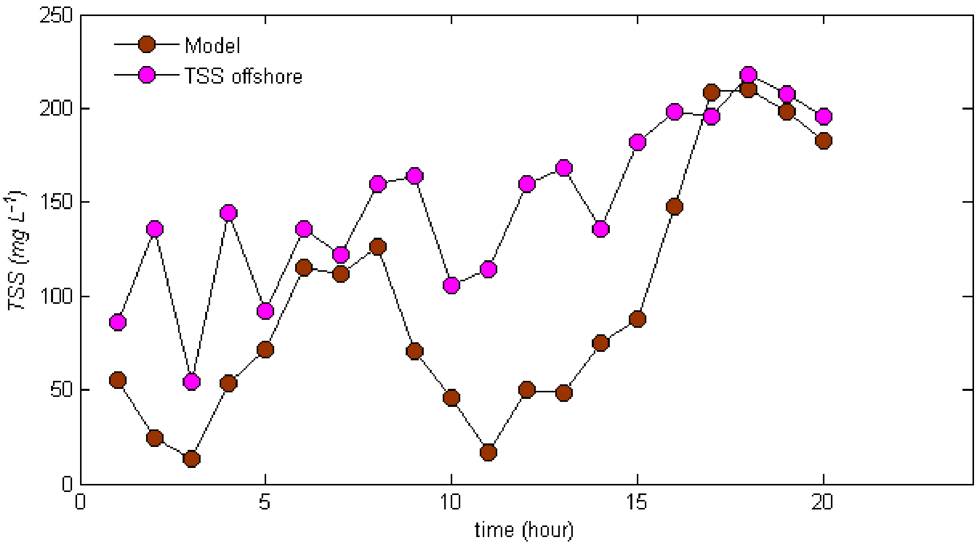

Figure 3. Suspended sediment transport of the surface in the mouth of the Dumai estuary over a period of one tidal cycle in the case of constant river discharge, brown is the model, and purple is the offshore data. The correlation value is about 0.9.

4. CONCLUSION

The dynamics of the estuary were expressed by the advection-diffusion equation, where the boundary conditions, distances, and interactions between the freshwater mass from the river and the saltwater mass from the sea remarkably determined the distribution process of suspended sediment transport. We investigated this process for areas close to the mouth of the estuary. Our measurement indicated that the advection process was clearly dominant. Although advection may be dominant, we still need to solve the advection-diffusion equation while taking into account the diffusion term. Furthermore, we applied an expansion multiple scale analysis, aiming to solve the advection-diffusion equation with the advection process being more dominant than the diffusion process. We obtained an analytical solution in terms of the Green's function. The results showed that in one tidal cycle, TSS was much higher on the surface, then it declined markedly after reaching the open sea. In addition, this current work found that the pattern of TSS distribution was in accordance with the constant river discharge and linear time-dependent river discharge. The model results obviously declared that TSS concentrations were higher offshore than near the mouth of the estuary, and this fully complied with the results of observation.

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APPENDIX
Solution of Eq. (18)

The solution with the Green's function is a problem of calculating the convolution integral. First, substituting Eq. (18) into Eq. (20), yields,

$$f\left(t_0,\chi,\zeta\right)=A_0 exp\left[\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(\zeta'\right)}d\zeta'\right]\left(D_x\frac{\partial^2\hat{F}(t_0,\chi)}{\partial\chi^2}-u(\chi,t_1)\frac{\partial\hat{F}(t_0,\chi)}{\partial\chi}\right) \tag{A1}$$

where,

$$\hat{F}(t_0,\chi)=exp\left[-\int_0^{t_0}\int_0^{\chi}\left(\frac{\Lambda}{U\left(\chi,t_0\right)-\omega}\right)d\chi' dt_0'\right] \tag{A2}$$

we note that a term containing the derivative t_1 will result in a second-order divisor in ε which will have a small value so that we ignore it. Substituting (A1) and Eq. (23) into Eq. (21), we arrive,

$$C^{(1)}\left(t_0,\chi,\zeta\right)=\frac{A_0}{2\sqrt{\pi}}\int_{-\zeta}^{0}exp\left(\int_{-\zeta}^{0}\frac{\Lambda}{w_s\left(\zeta-\zeta'\right)}d\zeta'+\int_{-\zeta'}^{0}\frac{\Lambda}{w_s\left(\zeta''\right)}d\zeta''\right)d\zeta'\times\left[\hat{G}_1\left(t_0,\chi\right)+\hat{G}_2\left(t_0,\chi\right)\right] \tag{A3}$$

where,

$$\hat{G}_1\left(t_0,\chi\right)=D_x\int_0^{t_0}\int_0^{\chi}\left\{exp\left[-\int_0^{t_0'}\int_0^{\chi'}\left(\frac{\Lambda}{U\left(\chi-\chi'',t_0-t_0'\right)-\omega}\right)dt_0''d\chi''\right]\frac{\partial^2\hat{F}(t_0',\chi')}{\partial\chi'^2}\right\}dt_0'd\chi' \tag{A4}$$

$$\hat{G}_2\left(t_0,\chi\right)=-\int_0^{t_0}\int_0^{\chi}U\left(\chi',t_1\right)\left\{exp\left[-\int_0^{t_0'}\int_0^{\chi'}\left(\frac{\Lambda}{U\left(\chi-\chi'',t_0-t_0'\right)-\omega}\right)dt_0''d\chi''\right]\frac{\partial\hat{F}(t_0',\chi')}{\partial\chi'}\right\}dt_0'd\chi' \tag{A5}$$

first, we calculate the derivation of $\hat{F}(t_0',\chi')$ as follow,

$$\frac{\partial\hat{F}(t_0',\chi')}{\partial\chi'}=-e^{-\hat{\Lambda}(t_0',\chi')}\frac{\partial\hat{\Lambda}\left(t_0',\chi'\right)}{\partial\chi'}=-e^{-\hat{\Lambda}(t_0',\chi')}\left[\int_0^{t_0'}\frac{\Lambda}{U\left(\chi',t_0''\right)-\omega}dt_0''\right] \tag{A6}$$

where,

$$\hat{\Lambda}\left(t_0',\chi'\right)=\int_0^{t_0'}\int_0^{\chi'}\left(\frac{\Lambda}{U\left(\chi'',t_0''\right)-\omega}\right)d\chi''dt_0'' \tag{A7}$$

and we have used the Leibniz's rule integration formula $\frac{d}{dx}\int_0^{g(x)}f(t)dt=f(g)g'(x)$. The second derivative,

$$\frac{\partial^2\hat{F}(t_0',\chi')}{\partial\chi'^2}=-e^{-\hat{\Lambda}(t_0',\chi')}\left[\left(\int_0^{t_0'}\frac{\Lambda}{U\left(\chi',t_0''\right)-\omega}dt_0''\right)^2-\int_0^{t_0'}\frac{\Lambda}{U\left(\chi',t_0''\right)-\omega}dt_0''\right] \tag{A8}$$

substituting A6 and A8 into A4 and A5, respectively, we get the complete solution of (A3) which is explicitly dependent on the form of $U(\chi,t)$.

$$U\left(\chi',t_0''\right)=A_1 cos\left(\chi',t_0''\right)-A_2\varepsilon cos2\left(\chi',t_0''\right)-A_3\varepsilon^2\chi' sin2\left(\chi',t_0''\right) \tag{A9}$$

where,

$$\left(\chi',t_0''\right)=k\chi'-\omega t_0'',\qquad A_1=\frac{gA}{c},\qquad A_2\varepsilon=\frac{g^2A^2}{8c^3},\qquad A_3\varepsilon^2=\frac{3g^2A^2\omega}{4c^4} \tag{A10}$$

we use Maclaurin's series ($\frac{1}{x}=\sum_{n=0}^{\infty}(1-x)^n$) and keep it up to second order. Then we will get,

$$\begin{aligned}&\left[A_1cos\left(\chi',t_0''\right)-A_2\varepsilon cos2\left(\chi',t_0''\right)-A_3\varepsilon^2\chi'sin2\left(\chi',t_0''\right)-\omega\right]^{-1}\approx\\&3A_1cos\left(\chi',t_0''\right)+2\omega A_1cos\left(\chi',t_0''\right)+4\omega+A_1^2cos^2\left(\chi',t_0''\right)+3A_2\varepsilon cos2\left(\chi',t_0''\right)\\&+2\omega A_2\varepsilon cos2\left(\chi',t_0''\right)+2A_1A_2\varepsilon cos\left(\chi',t_0''\right)cos2\left(\chi',t_0''\right)\\&+A_2^2\varepsilon^2cos^2 2\left(\chi',t_0''\right)+2A_1A_3\varepsilon^2cos\left(\chi',t_0''\right)\chi'sin2\left(\chi',t_0''\right)+3A_3\varepsilon^2\chi'sin2\left(\chi',t_0''\right)\end{aligned} \tag{A11}$$

thus we obtained,

$$\frac{\partial\hat{F}(t_0',\chi')}{\partial\chi'}=-e^{-\hat{\Lambda}(t_0',\chi')}\bar{F}(t_0',\chi');\qquad\frac{\partial^2\hat{F}(t_0',\chi')}{\partial\chi'^2}=-e^{-\hat{\Lambda}(t_0',\chi')}\left[\bar{F}^2(t_0',\chi')-\partial_{\chi'}\bar{F}(t_0',\chi')\right] \tag{A12}$$

where,

$$\begin{aligned}\bar{F}\left(t_0',\chi'\right)=&\left[\frac{3A_1}{\omega}sin(t_0',\chi')+2A_1sin(t_0',\chi')+4\omega t_0'+\frac{A_1^2}{2}\left(t_0'+\frac{1}{2\omega}sin2(t_0',\chi')\right)+\frac{3A_2\varepsilon}{\omega}sin2(t_0',\chi')\right.\\&+2A_2\varepsilon sin2(t_0',\chi')+\frac{2A_1A_2\varepsilon}{\omega}\left(sin(t_0',\chi')cos2(t_0',\chi')+\frac{4}{3}sin^3(t_0',\chi')\right)+A_2^2\varepsilon^2\left(\frac{t_0'}{2}+\frac{1}{8\omega}sin4(t_0',\chi')\right)\\&\left.-A_1A_2\varepsilon^2\left(\frac{1}{3\omega}cos3(t_0',\chi')+\frac{1}{\omega}cos(t_0',\chi')\right)-\frac{3A_3\chi'\varepsilon^2}{2\omega}cos2(t_0',\chi')\right]\end{aligned} \tag{A13}$$

and,

$$\begin{aligned}\frac{\hat{\Lambda}\left(t_0',\chi'\right)}{\Lambda}=&-\frac{3A_1}{k\omega}cos(t_0',\chi')-\frac{2A_1}{k}sin(t_0',\chi')-\frac{2A_1^2}{8k\omega}cos2(t_0',\chi')-\frac{3A_2\varepsilon}{2k\omega}cos2(t_0',\chi')-\frac{A_2\varepsilon}{k}cos2(t_0',\chi')\\&+\frac{A_1A_2\varepsilon}{k\omega}\left(-cos(t_0',\chi')-\frac{1}{3}cos3(t_0',\chi')+\frac{4}{9}cos^3(t_0',\chi')\right)-\frac{A_2^2\varepsilon^2}{32k\omega}cos4(t_0',\chi')+\frac{1}{2}\left(A_1^2+A_2^2\varepsilon^2\right)t_0'\chi'\\&-A_1A_2\varepsilon^2\left(\frac{1}{9k\omega}sin3(t_0',\chi')+\frac{1}{k\omega}cos(t_0',\chi')\right)-\frac{3A_3\chi'\varepsilon^2}{4k\omega}cos2(t_0',\chi')\end{aligned} \tag{A14}$$

by using an approximation $e^x\sim 1+x+\frac{1}{2!}x^2+\cdots$, and solving the integration with respect to (χ',t_0') in the Green's function Eq. (23), this leads,

$$\begin{aligned}&exp\left[-\int_0^{t_0'}\int_0^{\chi'}\left(\frac{\Lambda}{U\left(\chi-\chi'',t_0-t_0'\right)-\omega}\right)dt_0''d\chi''\right]\approx\Lambda-\frac{\left(3A_1+2\omega A_1\right)\Lambda}{k\omega}cos\left(\chi-\chi',t_0-t_0'\right)+2\omega\Lambda\chi' t_0'\\&+\frac{A_1^2}{4}\Lambda\left((\chi'-t_0')^2-\frac{1}{2k\omega}cos2\left(\chi-\chi',t_0-t_0'\right)\right)-\frac{\left(3A_2+2\omega A_2\right)\Lambda\varepsilon}{4k\omega}cos2\left(\chi-\chi',t_0-t_0'\right)+\ldots\end{aligned} \tag{A15}$$

further, substituting Eq (A15), (A14),(A13), and (A12) into Eq. (A4) and (A5), yields,

$$\frac{\hat{G}_1\left(t_0,\chi\right)}{D_x}=\iint_A d\chi'dt_0'+\iint_B d\chi'dt_0'+\iint_C d\chi'dt_0'+\iint_D d\chi'dt_0'+\ldots+\varepsilon\left[\iint_I d\chi'dt_0'+\iint_{II} d\chi'dt_0'+\iint_{III} d\chi'dt_0'+\iint_{IV} d\chi'dt_0'+\ldots\right] \tag{A16}$$

where,

$$\iint_A d\chi' dt_0' = \frac{27A_1^4(1+2\omega)\Lambda^2}{k^2\omega^3}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin^2\left(\chi', t_0'\right) cos\left(\chi', t_0'\right) d\chi' dt_0' \tag{A17}$$

$$\iint_B d\chi' dt_0' = \frac{18A_1^4(1+2\omega)\Lambda^2}{k^2\omega^2}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin^3\left(\chi', t_0'\right) d\chi' dt_0' \tag{A18}$$

$$\iint_C d\chi' dt_0' = \frac{12A_1^4(3+2\omega)\Lambda^2}{k^2\omega^3}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin^2\left(\chi', t_0'\right) cos2\left(\chi', t_0'\right) d\chi' dt_0' \tag{A19}$$

$$\iint_D d\chi' dt_0' = \frac{9A_1^4(3+2\omega)\Lambda^2}{k^2\omega^3}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin^2 2\left(\chi', t_0'\right) cos2\left(\chi', t_0'\right) d\chi' dt_0' \tag{A20}$$

$$\iint_I d\chi' dt_0' = \frac{27A_1^2 A_2(3+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin^2 2\left(\chi', t_0'\right) cos2\left(\chi', t_0'\right) d\chi' dt_0' \tag{A21}$$

$$\iint_{II} dx' dt_0' = \frac{18A_1^2 A_2(3+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^{x}\int_0^{t_0} cos\left(x - x'; t_0 - t_0'\right) sin^3 2\left(x', t_0'\right) cos2\left(x', t_0'\right) dx' dt_0' \tag{A22}$$

$$\iint_{III} dx' dt_0' = \frac{12A_1^2 A_2(1+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^{x}\int_0^{t_0} cos\left(x - x'; t_0 - t_0'\right) sin^2 2\left(x', t_0'\right) \sin\left(x', t_0'\right) dx' dt_0' \tag{A23}$$

$$\iint_{IV} dx' dt_0' = \frac{9A_1^2 A_2(3+2\omega)\Lambda^2}{4k^2\omega^3}\int_0^{x}\int_0^{t_0} cos\left(x - x'; t_0 - t_0'\right) sin^2 2\left(x', t_0'\right) sin2\left(x', t_0'\right) dx' dt_0' \tag{A24}$$

The other terms contain multiplication by A_2 squared or higher, where this term has a divisor of c3, so this can be seen as a small perturbation, and it can be ignored. By using the same procedure, we arrive,

$$\hat{G}_2\left(t_0, \chi\right) = \iint_{A'} d\chi' dt_0' + \iint_{B'} d\chi' dt_0' + \iint_{C'} d\chi' dt_0' + \ldots + \varepsilon\left[\iint_{I'} d\chi' dt_0' + \iint_{II'} d\chi' dt_0' + \iint_{III'} d\chi' dt_0' + \ldots\right] \tag{A25}$$

where,

$$\iint_{A'} d\chi' dt_0' = \frac{A_1^3(1+2\omega)\Lambda^2}{k\omega^3}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin\left(\chi', t_0'\right) cos\left(\chi', t_0'\right) d\chi' dt_0' \tag{A26}$$

$$\iint_{B'} d\chi' dt_0' = \frac{A_1^3(1+2\omega)\Lambda^2}{k\omega^3}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin\left(\chi', t_0'\right) cos^2\left(\chi', t_0'\right) d\chi' dt_0' \tag{A27}$$

$$\iint_{C'} d\chi' dt_0' = \frac{6A_1^3(1+2\omega)\Lambda^2}{k\omega}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin^2\left(\chi', t_0'\right) cos\left(\chi', t_0'\right) d\chi' dt_0' \tag{A28}$$

$$\iint_{I'} d\chi' dt_0' = \frac{6A_1^2 A_2(3+2\omega)\Lambda^2}{k\omega}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin\left(\chi', t_0'\right) cos2\left(\chi', t_0'\right) d\chi' dt_0' \tag{A29}$$

$$\iint_{II'} d\chi' dt_0' = -\frac{3A_1^2 A_2(3+2\omega)\Lambda^2}{k\omega}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin\left(\chi', t_0'\right) cos\left(\chi', t_0'\right) cos2\left(\chi', t_0'\right) d\chi' dt_0' \tag{A30}$$

$$\iint_{III'} d\chi' dt_0' = -\frac{6A_1^3 A_2(3+2\omega)\Lambda^2}{k\omega}\int_0^{\chi}\int_0^{t_0} cos\left(\chi - \chi'; t_0 - t_0'\right) sin^2\left(\chi', t_0'\right) cos\left(\chi', t_0'\right) cos2\left(\chi', t_0'\right) d\chi' dt_0' \tag{A31}$$

Finally, we use numerical integration to solve the convolution integral above.